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Stochastic formulation of quantum mechanics based on a complex Langevin equation

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Abstract. It is shown that a stochastic process described by a complex Langevin equation leads us to real-time quantum mechanics. We also derive a relation between a transition probability in our theory and a transition probability amplitude in real-time quantum mechanics in the framework of path-integral formulations. Finally, taking a harmonic oscillator case as an example, the Fokker-Planck equation is solved exactly, and a non-dissipation property of the stochastic process is pointed out.

1. Introduction

Formal similarities between quantum mechanics and the theory of stochastic processes have been pointed out and investigated by many authors [1] since the beginning of quantum mechanics. It is well known that the Schrödinger equation for a free particle has the same form as the diffusion equation for free Brownian motion, in the following way. The Schrödinger equation for a free particle is

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) \tag{1.1}$$

where m, $\hbar$ and $\Psi(x, t)$ are the particle mass, Planck constant and wavefunction, respectively. Through the replacement

$$\mathrm{i}t \to s \qquad \Psi(x, t) \to \Psi(x, s) \tag{1.2}$$

t and s being real-time and imaginary-time, respectively, (1.1) turns to the diffusion equation for free Brownian motion and diffusion constant $\alpha = \hbar/2m$:

$$\frac{\partial}{\partial s} \Psi(x, s) = \alpha \frac{\partial^2}{\partial x^2} \Psi(x, s). \tag{1.3}$$

As is also well known [2], the above analogy can be generalized to the case for a particle in a potential. Imaginary-time quantum mechanical motion of a particle in a potential can also be described by a Fokker-Planck equation, or by a corresponding Langevin equation. Thus imaginary-time quantum mechanics can be formulated within the stochastic-theoretical framework. This formulation seems to be useful for numerical simulations.

From the fundamental point of view, however, it would be important to examine whether real-time quantum mechanics can also be formulated within the stochastic-theoretical framework. This is the task of the present paper. At first sight this seems

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to be not so easy, because a straightforward application of the manipulation of the stochastic formulation of imaginary-time quantum mechanics to real-time quantum mechanics leads us to a complex Fokker–Planck function. Such a function cannot be explained as a probability distribution. In this paper, we overcome this difficulty by means of a complex Langevin equation which yields a real positive probability distribution.

This paper is organized as follows. In section 2, for convenience for later discussions, we briefly review the stochastic formulation of imaginary-time quantum mechanics. In section 3, we present such a complex Langevin equation and show that it leads us to the real-time Schrödinger equation. In section 4, we derive a relation between a transition probability in our theory and a transition probability amplitude in real-time quantum mechanics in the framework of path-integral formulations of both theories. In section 5, taking a harmonic oscillator case as an example, the Fokker–Planck equation is solved exactly, and a non-dissipation property of the stochastic process is pointed out. Section 6 is devoted to concluding remarks.

2. Stochastic formulation of imaginary-time quantum mechanics

The Schrödinger equation for a particle in a potential $V(x)$ is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(x,t)=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right)\Psi(x,t). \tag{2.1}$$

Through the replacement (1.2), (2.1) turns to the imaginary-time Schrödinger equation

$$\hbar\frac{\partial}{\partial s}\Psi(x,s)=\left(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-V(x)\right)\Psi(x,s). \tag{2.2}$$

Let us introduce a function, $W(x)$, which is a solution of the following Riccati-type differential equation:

$$V(x)=\frac{m}{2}\left[\left(\frac{\partial W(x)}{\partial x}\right)^2-\frac{\hbar}{m}\frac{\partial^2 W(x)}{\partial x^2}\right]+E \tag{2.3}$$

where E is a suitable constant which will be determined later. By the transformation

$$W(x)=-\frac{\hbar}{m}\ln\varphi(x) \tag{2.4}$$

(2.3) becomes the quantum mechanical Hamiltonian eigenequation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right)\varphi(x)=E\varphi(x). \tag{2.5}$$

Hereafter, we shall require $W(x)$ to be a real function. Then, $\varphi(x)>0$ for all x. This means that the only permissible solution of (2.5) is the ground state; $\varphi(x)=\varphi_0(x)$, $E=E_0$. Substituting (2.3) into (2.2), we have

$$\hbar\frac{\partial}{\partial s}\Psi(x,s)=\left\{\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\frac{m}{2}\left[\left(\frac{\partial W(x)}{\partial x}\right)^2-\frac{\hbar}{m}\frac{\partial^2 W(x)}{\partial x^2}\right]-E\right\}\Psi(x,s). \tag{2.6}$$

Further introducing a function, $P(x,s)$, defined by

$$P(x,s)\equiv\exp\left(\frac{1}{\hbar}(Es-mW(x))\right)\Psi(x,s) \tag{2.7}$$

we see that $P(x, s)$ obeys

$$\frac{\partial}{\partial s} P(x, s) = \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} P(x, s) + \frac{\partial}{\partial x}\left(\frac{\partial W(x)}{\partial x} P(x, s)\right) \tag{2.8}$$

which is a Fokker–Planck equation for a stochastic process with drift velocity $\partial W(x)/\partial x$ and diffusion constant $\hbar/2m$. This stochastic process can also be described by a stochastic differential equation, i.e. a Langevin equation given in the form

$$\frac{\mathrm{d}}{\mathrm{d}s} x(s) = -\left.\frac{\partial W(x)}{\partial x}\right|_{x \to x(s)} + \eta(s) \tag{2.9a}$$

where $\eta(s)$ is a Gaussian white noise which satisfies the following statistical properties:

$$\langle \eta(s) \rangle = 0 \tag{2.9b}$$

$$\langle \eta(s)\eta(s') \rangle = \frac{\hbar}{m}\,\delta(s - s'). \tag{2.9c}$$

If the Schrödinger equation (2.1) gives a finite-energy ground state which is normalizable and non-degenerate, the above stochastic process gives a thermal equilibrium state at the limit $s \to \infty$. The n-point stochastic correlation function at this thermal equilibrium coincides with the n-point ground-state-to-ground-state Green function in imaginary-time quantum mechanics. Actually, Schneider *et al* [2] calculated this Green function by a numerical simulation based on the Langevin equation (2.9).

3. Stochastic formulation of real-time quantum mechanics based on a complex Langevin equation

In order to make a transition from the imaginary-time formulation reviewed in the preceding section to the real-time formulation, first we introduce the replacement

$$s \to \mathrm{i}t \qquad x(s) \to x(t) \qquad \eta(s) \to -\mathrm{i}\eta(t). \tag{3.1}$$

Then, the imaginary-time Langevin equation (2.9) turns to the form

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = -\mathrm{i}\left.\frac{\partial W(x)}{\partial x}\right|_{x \to x(t)} + \eta(t) \tag{3.2a}$$

$$\langle \eta(t) \rangle = 0 \tag{3.2b}$$

$$\langle \eta(t)\eta(t') \rangle = \frac{\mathrm{i}\hbar}{m}\,\delta(t - t'). \tag{3.2c}$$

For derivation of (3.2c) from (2.9c), we have used the formal relation $\delta[\mathrm{i}(t-t')] = (1/\mathrm{i})\delta(t-t')$.

To make sense of (3.2), we must consider $x(t)$ and $\eta(t)$ to be complex variables because of the imaginary coefficients in (3.2). Then, we must treat (3.2) as a Langevin equation for real and imaginary parts of $x(t)$. Such an equation is called the *complex Langevin equation.* Methods of the complex Langevin equation, which we use throughout this paper, have been investigated and developed in relation to the Minkowski formulation of Parisi and Wu's stochastic quantization [4] and numerical simulations of complex systems [5].

In (3.2a), $W(x)$ is also determined by the Riccati-type differential equation (2.3). In the imaginary-time formulation, we have required $W(x)$ to be a real function. In the real-time formulation, $W(x)$ is not necessarily a real function because (3.2) is a complex-valued equation. In this paper, however, we shall successively require $W(x)$ to be a real function for simplicity. Notice that in (3.2a) $\partial W(x)/\partial x|_{x\to x(t)}$ is complex although $W(x)$ is real.

One of the simplest ways to define the complex random variable $\eta(t)$ satisfying (3.2b, c) in terms of real Gaussian white noises is

$$\eta(t) \equiv \sqrt{\frac{\hbar}{2m}}[(\eta_1(t)-\eta_2(t))+\mathrm{i}(\eta_1(t)+\eta_2(t))] \tag{3.3a}$$

where $\eta_1(t)$ and $\eta_2(t)$ are real Gaussian white noises satisfying

$$\langle\eta_1(t)\rangle=\langle\eta_2(t)\rangle=0 \tag{3.3b}$$

$$\langle\eta_1(t)\eta_1(t')\rangle=A\delta(t-t') \tag{3.3c}$$

$$\langle\eta_2(t)\eta_2(t')\rangle=B\delta(t-t') \tag{3.3d}$$

$$\langle\eta_1(t)\eta_2(t')\rangle=0 \tag{3.3e}$$

$$A-B=1 \qquad A>0 \qquad B>0. \tag{3.3f}$$

Hereafter, we use the following notation:

$$x_{\mathrm{R}}(t)\equiv \mathrm{Re}(x(t)) \qquad x_{\mathrm{I}}(t)\equiv \mathrm{Im}(x(t)) \tag{3.4a}$$

$$\eta_{\mathrm{R}}(t)\equiv \mathrm{Re}(\eta(t)) \qquad \eta_{\mathrm{I}}(t)\equiv \mathrm{Im}(\eta(t)) \tag{3.4b}$$

$$\begin{aligned} R(t) &\equiv R(x_{\mathrm{R}}(t), x_{\mathrm{I}}(t)) \equiv \mathrm{Re}\left(\mathrm{i}\frac{\partial W(x)}{\partial x}\bigg|_{x\to x(t)}\right) \\ I(t) &\equiv I(x_{\mathrm{R}}(t), x_{\mathrm{I}}(t)) \equiv \mathrm{Im}\left(\mathrm{i}\frac{\partial W(x)}{\partial x}\bigg|_{x\to x(t)}\right). \end{aligned} \tag{3.4c}$$

The equation (3.2a) is now decomposed into real and imaginary parts, as follows,

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{R}}(t)=-R(t)+\eta_{\mathrm{R}}(t) \tag{3.5a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_{\mathrm{I}}(t)=-I(t)+\eta_{\mathrm{I}}(t) \tag{3.5b}$$

where $\eta_{\mathrm{R}}(t)$ and $\eta_{\mathrm{I}}(t)$ satisfy the following statistical properties:

$$\langle\eta_{\mathrm{R}}(t)\rangle=\langle\eta_{\mathrm{I}}(t)\rangle=0 \tag{3.5c}$$

$$\langle\eta_{\mathrm{R}}(t)\eta_{\mathrm{R}}(t')\rangle=\frac{\hbar}{2m}(A+B)\delta(t-t') \tag{3.5d}$$

$$\langle\eta_{\mathrm{I}}(t)\eta_{\mathrm{I}}(t')\rangle=\frac{\hbar}{2m}(A+B)\delta(t-t') \tag{3.5e}$$

$$\langle\eta_{\mathrm{R}}(t)\eta_{\mathrm{I}}(t')\rangle=\frac{\hbar}{2m}\delta(t-t'). \tag{3.5f}$$

It follows from the general theory of stochastic processes that the stochastic process given by (3.5) is also described by a real positive probability distribution, $P(x_R, x_I, t)$, which obeys the following Fokker–Planck equation:

$$\frac{\partial}{\partial t} P(x_R, x_I, t)$$
$$= \frac{\hbar}{4m}\left((A+B)\frac{\partial^2}{\partial x_R^2} + 2\frac{\partial^2}{\partial x_R \partial x_I} + (A+B)\frac{\partial^2}{\partial x_I^2}\right) P(x_R, x_I, t)$$
$$+ \frac{\partial}{\partial x_R}(R(x_R, x_I)P(x_R, x_I, t)) + \frac{\partial}{\partial x_I}(I(x_R, x_I)P(x_R, x_I, t)). \qquad (3.6)$$

Next, we shall show that (3.6) leads us to the real-time Schrödinger equation. For the purpose of this, we introduce the *effective Fokker–Planck distribution*, $P_{\text{eff}}(x_R, t)$, defined by

$$P_{\text{eff}}(x_R, t) \equiv \int_{-\infty}^{+\infty} dx_I \, P(x_R - ix_I, x_I, t). \qquad (3.7)$$

Notice that $P_{\text{eff}}(x_R, t)$ is complex although $P(x_R, x_I, t)$ is real.

Operating a translation operator, $\exp[-ix_I(\partial/\partial x_R)]$, on (3.6) from the left, and performing an integration with respect to x_I from $-\infty$ to $+\infty$, we obtain the *effective Fokker–Planck equation*

$$\frac{\partial}{\partial t} P_{\text{eff}}(x_R, t) = i\left[\frac{\hbar}{2m}\frac{\partial^2}{\partial x_R^2} P_{\text{eff}}(x_R, t) + \frac{\partial}{\partial x_R}\left(\frac{\partial W(x_R)}{\partial x_R} P_{\text{eff}}(x_R, t)\right)\right]. \qquad (3.8)$$

For the derivation of (3.8), we have used the following formulae

$$\exp\left(-ix_I\frac{\partial}{\partial x_R}\right) x_R \exp\left(ix_I\frac{\partial}{\partial x_R}\right) = x_R - ix_I \qquad (3.9a)$$

$$\exp\left(-ix_I\frac{\partial}{\partial x_R}\right) x_I \exp\left(ix_I\frac{\partial}{\partial x_R}\right) = x_I \qquad (3.9b)$$

$$\exp\left(-ix_I\frac{\partial}{\partial x_R}\right) \frac{\partial}{\partial x_R} \exp\left(ix_I\frac{\partial}{\partial x_R}\right) = \frac{\partial}{\partial x_R} \qquad (3.9c)$$

$$\exp\left(-ix_I\frac{\partial}{\partial x_R}\right) \frac{\partial}{\partial x_I} \exp\left(ix_I\frac{\partial}{\partial x_R}\right) = \frac{\partial}{\partial x_I} + i\frac{\partial}{\partial x_R} \qquad (3.9d)$$

and assumed that surface terms caused by the integration vanish.

Further introducing a function, $\Psi(x_R, t)$, defined by

$$\Psi(x_R, t) \equiv \exp\left(-\frac{1}{\hbar}(iEt - mW(x_R))\right) P_{\text{eff}}(x_R, t) \qquad (3.10)$$

we see that $\Psi(x_R, t)$ obeys the real-time Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x_R, t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_R^2} + V(x_R)\right)\Psi(x_R, t) \qquad (3.11)$$

where (2.3) has been used.

We have thus shown that the stochastic process described by the Langevin equation (3.5) leads us to real-time quantum mechanics.

4. Path-integral formulation

In this section, we shall derive a relation between a transition probability in our theory and a transition probability amplitude in real-time quantum mechanics. For such a derivation, it will be convenient to work in path-integral formulations of both theories.

Let $T(x''_R, x''_I, t''|x'_R, x'_I, t')$ be a probability in our theory for a transition from $(x_R, x_I)=(x'_R, x'_I)$ at time t' to $(x_R, x_I)=(x''_R, x''_I)$ at time t''. Following the general theory of stochastic processes, we can write this probability amplitude in the Wiener-Onsager-Machlup path-integral representation [6], as follows,

$$T(x''_R, x''_I, t''|x'_R, x'_I, t) = N \int_{(x_R(t'),x_I(t'))=(x'_R,x'_I)}^{(x_R(t''),x_I(t''))=(x''_R,x''_I)} \mathscr{D}x_R\, \mathscr{D}x_I \times \exp\left(-\int_{t'}^{t''} L_{FP}(x_R(t), x_I(t), \dot{x}_R(t), \dot{x}_I(t))\, dt \right) \tag{4.1a}$$

where

$$L_{FP}(x_R(t), x_I(t), \dot{x}_R(t), \dot{x}_I(t)) \equiv \frac{m}{4\hbar}\left[\frac{1}{A}\left(\frac{dx_R(t)}{dt}+R(t)+\frac{dx_I(t)}{dt}+I(t)\right)^2 + \frac{1}{B}\left(\frac{dx_R(t)}{dt}+R(t)-\frac{dx_I(t)}{dt}-I(t)\right)^2\right]. \tag{4.1b}$$

In (4.1) and in the following, N represents a proper normalization constant in each formula.

It is more convenient for later calculations to rewrite (4.1) in the discretized form

$$T(x''_R, x''_I, t''|x'_R, x'_I, t') = N \lim_{n\to+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n-1} dx_R(t_i)\, dx_I(t_i) \times \exp\left(-\sum_{i=1}^{n} \tilde{L}_{FP}(x_R(t_i), x_I(t_i), x_R(t_{i-1}), x_I(t_{i-1}))\Delta t \right) \tag{4.2a}$$

where

$$\tilde{L}_{FP}(x_R(t_i), x_I(t_i), x_R(t_{i-1}), x_I(t_{i-1})) \equiv \frac{m}{4\hbar}\left[\frac{1}{A}\left(\frac{x_R(t_i)-x_R(t_{i-1})}{\Delta t}+R(t_{i-1})+\frac{x_I(t_i)-x_I(t_{i-1})}{\Delta t}+I(t_{i-1})\right)^2 + \frac{1}{B}\left(\frac{x_R(t_i)-x_R(t_{i-1})}{\Delta t}+R(t_{i-1})-\frac{x_I(t_i)-x_I(t_{i-1})}{\Delta t}-I(t_{i-1})\right)^2\right] \tag{4.2b}$$

with

$$t_0=t' \qquad t_n=t'' \qquad \Delta t=\frac{t_n-t_0}{n} \qquad t_i=t_0+i\Delta t$$
$$x_R(t_0)=x'_R \qquad x_I(t_0)=x'_I \qquad x_R(t_n)=x''_R \qquad x_I(t_n)=x''_I. \tag{4.2c}$$

We can rewrite (4.2a), as follows,

$$
\begin{aligned}
&T(x_R'', x_I'', t''|x_R', x_I', t')\\
&\quad = N \lim_{n\to+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^{n-1} \mathrm{d}x_R(t_i)\,\mathrm{d}x_I(t_i)\\
&\quad\times \prod_{i=0}^{n} \exp\left(\mathrm{i}x_I(t_i)\frac{\partial}{\partial x_R(t_i)}\right)\exp\left(-\mathrm{i}x_I(t_i)\frac{\partial}{\partial x_R(t_i)}\right)\\
&\quad\times \exp\left(-\sum_{i=1}^{n} \tilde{L}_{FP}(x_R(t_i), x_I(t_i), x_R(t_{i-1}), x_I(t_{i-1}))\Delta t\right)\\
&\quad = N \lim_{n\to+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^{n-1} \mathrm{d}x_R(t_i)\,\mathrm{d}x_I(t_i)\\
&\quad\times \exp\left(\mathrm{i}x_I''\frac{\partial}{\partial x_R''}+\mathrm{i}x_I'\frac{\partial}{\partial x_R'}\right)\exp\left(-\mathrm{i}x_I''\frac{\partial}{\partial x_R''}-\mathrm{i}x_I'\frac{\partial}{\partial x_R'}\right)\\
&\quad\times \prod_{i=1}^{n-1}\left[\exp\left(-\mathrm{i}x_I(t_i)\frac{\partial}{\partial x_R(t_i)}\right)\times 1\right]\exp\left(-\mathrm{i}x_I(t_i)\frac{\partial}{\partial x_R(t_i)}\right)\\
&\quad\times \exp\left(-\sum_{i=1}^{n} \tilde{L}_{FP}(x_R(t_i), x_I(t_i), x_R(t_{i-1}), x_I(t_{i-1}))\Delta t\right)\\
&\quad = N \lim_{n\to+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^{n-1} \mathrm{d}x_R(t_i)\,\mathrm{d}x_I(t_i)\exp\left(\mathrm{i}x_I''\frac{\partial}{\partial x_R''}+\mathrm{i}x_I'\frac{\partial}{\partial x_R'}\right)\\
&\quad\times \exp\left(-\sum_{i=1}^{n} \tilde{L}_{FP}(x_R(t_i)-\mathrm{i}x_I(t_i), x_I(t_i), x_R(t_{i-1})-\mathrm{i}x_I(t_{i-1}), x_I(t_{i-1}))\Delta t\right).
\end{aligned}\tag{4.3}
$$

In (4.3), the second equality is obtained by performing partial integrations with respect to $x_R(t_i)$ $(i=1,2,3,\dots,n-1)$ where we assume that surface terms vanish. After a short calculation, we have

$$
\begin{aligned}
&\tilde{L}_{FP}(x_R(t_i)-\mathrm{i}x_I(t_i), x_I(t_i), x_R(t_{i-1})-\mathrm{i}x_I(t_{i-1}), x_I(t_{i-1}))\\
&\quad = \frac{\mathrm{i}m}{2\hbar AB}\left(\frac{x_I(t_i)-x_I(t_{i-1})}{\Delta t}+F(x_R(t_i), x_R(t_{i-1}), x_I(t_{i-1}))\right)^2\\
&\quad -\frac{\mathrm{i}m}{2\hbar}\left(\frac{x_R(t_i)-x_R(t_{i-1})}{\Delta t}+\mathrm{i}\frac{\partial W(x_R(t_{i-1}))}{\partial x_R(t_{i-1})}\right)^2
\end{aligned}\tag{4.4}
$$

where F is a function of $x_R(t_i)$, $x_R(t_{i-1})$, $x_I(t_{i-1})$ and independent of $x_I(t_i)$. Operating $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}x_I'\,\mathrm{d}x_I''\exp[-\mathrm{i}x_I''(\partial/\partial x_R'')-\mathrm{i}x_I'(\partial/\partial x_R')]$ on (4.3) from the left, we have

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}x_I'\,\mathrm{d}x_I''\,T(x_R''-\mathrm{i}x_I'', x_I'', t''|x_R'-\mathrm{i}x_I', x_I', t')\\
&\quad = N \lim_{N\to+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{i=1}^{n-1} \mathrm{d}x_R(t_i) \prod_{i=0}^{n}\mathrm{d}x_I(t_i)\\
&\quad\times \exp\left(-\sum_{i=1}^{n} \tilde{L}_{FP}(x_R(t_i)-\mathrm{i}x_I(t_i), x_I(t_i), x_R(t_{i-1})-\mathrm{i}x_I(t_{i-1}), x_I(t_{i-1}))\Delta t\right).
\end{aligned}\tag{4.5}
$$

We can then easily perform integrations with respect to $x_I(t_n), x_I(t_{n-1}), x_I(t_{n-2}), \ldots, x_I(t_0)$ in this order (these are ill-defined Gaussian integrations, but we formally perform them as we usually do in the theory of Feynman path-integral in Minkowski space), to obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}x_I'\,\mathrm{d}x_I''\, T(x_R''-\mathrm{i}x_I'', x_I'', t''|x_R'-\mathrm{i}x_I', x_I', t')$$
$$= N\int_{x_R(t')=x_R'}^{x_R(t'')=x_R''} \mathscr{D}x_R \exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left(\frac{\mathrm{d}x_R(t)}{\mathrm{d}t}+\mathrm{i}\frac{\partial W(x_R)}{\partial x_R}\bigg|_{x_R\to x_R(t)}\right)^2 \mathrm{d}t\right]. \tag{4.6}$$

Following the discussions on the relation between two kinds of path-integral formulations of Parisi and Wu's stochastic quantization [7], we can replace the exponent of the right-hand side of (4.6) with

$$\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left[\left(\frac{\mathrm{d}x_R(t)}{\mathrm{d}t}\right)^2+2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}[W(x_R(t))]-\left(\frac{\partial W(x_R)}{\partial x_R}\bigg|_{x_R\to x_R(t)}\right)^2\right.$$
$$\left.+\frac{\hbar}{m}\frac{\partial^2 W(x_R)}{\partial x_R^2}\bigg|_{x_R\to x_R(t)}\right]\mathrm{d}t.$$

Then, we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}x_I'\,\mathrm{d}x_I''\, T(x_R''-\mathrm{i}x_I'', x_I'', t''|x_R'-\mathrm{i}x_I', x_I', t')$$
$$= N\exp\left(-\frac{m}{\hbar}[W(x_R(t''))-W(x_R(t'))]\right)\int_{x_R(t')=x_R'}^{x_R(t'')=x_R''}\mathscr{D}x_R$$
$$\times\exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left[\left(\frac{\mathrm{d}x_R(t)}{\mathrm{d}t}\right)^2-\left(\frac{\partial W(x_R)}{\partial x_R}\bigg|_{x_R\to x_R(t)}\right)^2\right.\right.$$
$$\left.\left.+\frac{\hbar}{m}\frac{\partial^2 W(x_R)}{\partial x_R^2}\bigg|_{x_R\to x_R(t)}\right]\mathrm{d}t\right\}. \tag{4.7}$$

Then, using (2.3), we finally have the following relation

$$K(x_R'', t''|x_R', t')$$
$$= \exp\left(\frac{m}{\hbar}[W(x_R(t''))-W(x_R(t'))]-\frac{\mathrm{i}}{\hbar}E(t''-t')\right)$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}x_I'\,\mathrm{d}x_I''\, T(x_R''-\mathrm{i}x_I'', x_I'', t''|x_R'-\mathrm{i}x_I', x_I', t') \tag{4.8a}$$

where $K(x_R'', t''|x_R', t')$ is a probability amplitude in real-time quantum mechanics for a transition from $x_R=x_R'$ at time t' to $x_R=x_R''$ at time t'', given in the Feynman path-integral representation as follows,

$$K(x_R'', t''|x_R', t')$$
$$\equiv N\int_{x_R(t')=x_R'}^{x_R(t'')=x_R''}\mathscr{D}x_R\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\left(\frac{\mathrm{d}x_R(t)}{\mathrm{d}t}\right)^2-V(x_R(t))\right]\mathrm{d}t\right\}. \tag{4.8b}$$

We have thus derived a relation between a transition probability in our theory and a transition probability amplitude in real-time quantum mechanics.

5. An example: a harmonic oscillator case

In this section, in order to illustrate the validities of results of the discussions in the preceding sections and to point out a special property of our stochastic process, we consider a harmonic oscillator as an example, for which everything is exactly solvable.

Consider the potential for a harmonic oscillator:

$$V(x)=\tfrac{1}{2}m\omega^2x^2. \tag{5.1}$$

In this case, the Riccati equation (2.3) can be easily solved, to give

$$W(x)=\tfrac{1}{2}\omega x^2 \tag{5.2a}$$

$$E=\tfrac{1}{2}\hbar\omega. \tag{5.2b}$$

The Fokker-Planck equation (3.6) then becomes

$$\frac{\partial}{\partial t}P(x_R,x_I,t)=\hat{H}P(x_R,x_I,t) \tag{5.3a}$$

where

$$\hat{H}\equiv\frac{\hbar}{4m}\left((A+B)\frac{\partial^2}{\partial x_R^2}+2\frac{\partial^2}{\partial x_R\partial x_I}+(A+B)\frac{\partial^2}{\partial x_I^2}\right)$$
$$-\frac{\partial}{\partial x_R}(\omega x_I)+\frac{\partial}{\partial x_I}(\omega x_R). \tag{5.3b}$$

With an initial condition $P(x_R,x_I,t')=P_{t'}(x_R,x_I)$, we can rewrite (5.3), as follows,

$$P(x_R,x_I,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}x'_R\,\mathrm{d}x'_I\;T(x_R,x_I,t|x'_R,x'_I,t')P_{t'}(x'_R,x'_I) \tag{5.4a}$$

$$(D_t-\hat{H})T(x_R,x_I,t|x'_R,x'_I,t')=\delta(x_R-x'_R)\delta(x_I-x'_I)\delta(t-t') \tag{5.4b}$$

where $T(x_R,x_I,t|x'_R,x'_I,t')$ is the transition probability discussed in the preceding section.

Using well known procedures [8], we have a solution of (5.4*b*):

$$T(x_R,x_I,t|x'_R,x'_I,t')$$
$$=\frac{1}{4\pi^2}\theta(T)\frac{1}{\sqrt{QS-R^2}}\exp\left(-\frac{Sa^2-2Rab+Qb^2}{QS-R^2}\right) \tag{5.5a}$$

where

$$Q\equiv\frac{\hbar}{4m}\left((A+B)T+\frac{1}{\omega}\sin^2(\omega T)\right) \tag{5.5b}$$

$$R\equiv\frac{\hbar}{4m\omega}\sin(\omega T)\cos(\omega T) \tag{5.5c}$$

$$S\equiv\frac{\hbar}{4m}\left((A+B)T-\frac{1}{\omega}\sin^2(\omega T)\right) \tag{5.5d}$$

$$a\equiv\tfrac{1}{2}[x_R-\cos(\omega T)x'_R-\sin(\omega T)x'_I] \tag{5.5e}$$

$$b\equiv\tfrac{1}{2}[x_I+\sin(\omega T)x'_R-\cos(\omega T)x'_I] \tag{5.5f}$$

with $T=t-t'$.

Substituting (5.5) into the right-hand side of (4.8a), and performing a short calculation, we have

$$K(x_{\rm R}, t|x'_{\rm R}, t') = N\theta(T)\left(\frac{m\omega}{2\pi {\rm i}\hbar \sin(\omega T)}\right)^{1/2} \times \exp\left(\frac{{\rm i}m\omega}{2\hbar \sin(\omega T)}[(x_{\rm R}^2 + x_{\rm R}'^2)\cos(\omega T) - 2x_{\rm R}x'_{\rm R}]\right). \tag{5.6}$$

The right-hand side is a well known form of quantum mechanical transition probability amplitude for a harmonic oscillator.

The stochastic process defined by the Fokker–Planck equation (5.3) does not have a thermal equilibrium state as is different from cases in the imaginary-time formulation mentioned in section 2. To see this, consider the corresponding Langevin equation, written in the form

$$\frac{\rm d}{{\rm d}t}\begin{pmatrix} x_{\rm R}(t) \\ x_{\rm I}(t) \end{pmatrix} = A\begin{pmatrix} x_{\rm R}(t) \\ x_{\rm I}(t) \end{pmatrix} + \begin{pmatrix} \eta_{\rm R}(t) \\ \eta_{\rm I}(t) \end{pmatrix} \tag{5.7a}$$

where

$$A \equiv \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \tag{5.7b}$$

and $\eta_{\rm R}(t)$ and $\eta_{\rm I}(t)$ are random variables satisfying (3.5c–f). Dissipation behaviours of stochastic processes defined by Langevin equations of the type of (5.7a) are determined by real parts of eignevalues of As. In the present case, eigenvalues of A are $\pm {\rm i}\omega$, i.e. pure imaginary. Therefore, this process is not a dissipation process and does not have a thermal equilibrium state. This reflects the time reversibility in real-time quantum mechanics.

6. Conclusions

We have shown that a stochastic process described by the Langevin equation (3.5) leads us to real-time quantum mechanics. It should be confirmed that a probability distribution in our theory, which is governed by the Fokker–Planck equation (3.6), is real positive. We have also derived a relation between a transition probability in our theory and a transition probability amplitude in real-time quantum mechanics in path-integral formulations. The relation is given by the formula (4.8a).

Finally, taking a harmonic oscillator case, we have illustrated validities of our theory by solving the Fokker–Planck equation exactly, and pointed out a non-dissipation property of the stochastic process.

We have used the terminology 'probability distribution' for $P(x_{\rm R}, x_{\rm I}, t)$ only for convenience. In fact, $P(x_{\rm R}, x_{\rm I}, t)$ does not denote a physically realistic probability distribution. In that respect, our approach is different from ones by Bohm [9] or Nelson [10].

Our theory offers alternative calculation methods based on classical stochastic mechanics. Actually, once $W(x)$ is known, we can describe real-time quantum mechanical evolutions by the Langevin equation (3.5). Then, it is expected that we can solve real-time quantum mechanical problems by means of Monte Carlo simulations based on this Langevin equation. Furthermore, our stochastic-theoretical approach provides us with an intuitive picture for quantum mechanics, so that it might lead us to further insights into foundations of quantum mechanics.

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